

RECTIFIABILITY OF SETS OF FINITE PERIMETER IN A CLASS OF CARNOT GROUPS OF ARBITRARY STEP

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ABSTRACT. In this Note, we define a class of Carnot groups of arbitrary step (that are called “groups of type \star ” throughout the paper), and we prove that the reduced boundary of a set of finite intrinsic perimeter in a group of type \star is rectifiable in the intrinsic sense (De Giorgi’s rectifiability theorem). This result extends previous results proved by Franchi, Serapioni & Serra Cassano in step 2 groups.

1. INTRODUCTION

This article is intended to extend rectifiability and divergence theorems regarding step 2 Carnot groups ([11], Theorem 3.9 and Theorem 3.10), to a larger class of Carnot groups.

In order to illustrate our results, let us start by giving a gist of the background (for an exhaustive presentation, we refer for instance to [6], [24] and to the recent survey [2]).

In his celebrated paper [5], De Giorgi showed the link between Caccioppoli’s perimeter theory and those of Carathéodory, Hausdorff, and Federer, as well as with that of rectifiable sets, by proving that, for any subset $E \subset \mathbb{R}^n$ of locally finite perimeter, the perimeter measure is concentrated on a subset of ∂E , the so-called *reduced boundary*, that is $(n-1)$ -rectifiable, i.e., up to a set of vanishing $(n-1)$ -dimensional Hausdorff measure, it can be written as a countable union of compact portions of \mathbf{C}^1 -hypersurfaces. In fact, a crucial step of De Giorgi’s proof consists in a blow-up theorem, i.e. in proving that at any point x of the reduced boundary, when $r \rightarrow \infty$ the sets $E_r := r(E - x)$ tend locally in measure to a half-space.

On the other hand, the notion of perimeter can be formulated in a very general setting in metric spaces ([1], [18]), and, in particular, in Carnot groups (connected, simply connected, stratified nilpotent Lie groups). Detailed definitions are given below: here we restrict ourselves to remind that a Lie group \mathbb{G} is called a Carnot group if its Lie algebra \mathfrak{g} admits the stratification $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$, with $[V_1, V_i] = V_{i+1}$, where $[V_1, V_i]$ is the subspace of \mathfrak{g} generated by commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. The integer k is said the step of the group.

In 2001 Franchi, Serapioni & Serra Cassano ([10]) extended De Giorgi’s theorem to Heisenberg groups, and then, in 2003, to Carnot groups of step $k = 2$ ([11]), proving also that the blow-up theorem may fail to hold in groups of step $k > 2$.

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In Carnot groups of arbitrary step, only a partial result is known, that has been proved in [3] by Ambrosio, Kleiner & Le Donne. In fact, the authors show that, given a set E of locally finite perimeter in a Carnot group \mathbb{G} , then for almost every $x \in \mathbb{G}$ (with respect to the perimeter measure of E), some blow-up of E at x is a vertical halfspace.

The class of Carnot groups we define in this Note can be easily described as follows: if \mathbb{G} is a Carnot group with stratified Lie algebra $\mathfrak{g} = V_1 \oplus \dots \oplus V_k$, we assume that there exists a basis (X_1, \dots, X_{m_1}) of V_1 such that

$$[X_j, [X_j, X_i]] = 0 \text{ for } i, j = 1, \dots, m_1$$

We shall say that these groups are of type \star .

Obviously, step 2 Carnot groups are of type \star (in particular, Heisenberg groups are of type \star). If $m \in \mathbb{N}$, other important non-trivial examples are the Lie groups of unit upper triangular $(m+1) \times (m+1)$ -matrices with 1's on the diagonal: these groups are the nilpotent groups that comes from the Iwasawa decomposition of $GL_{m+1}(\mathbb{R})$ (see Example 2.3). In particular, this shows that there exist Carnot groups of type \star of arbitrarily large step.

The core of this Note can be summarised as follows (see Theorems 2.10, 2.11 and 2.12 for precise statements):

If $E \subseteq \mathbb{G}$ is a \mathbb{G} -Caccioppoli set in Carnot group \mathbb{G} of type \star , then

$$\partial_{\mathbb{G}}^* E \text{ is } (Q-1)\text{-dimensional } \mathbb{G}\text{-rectifiable.}$$

Let us give a sketch of the content of the present article.

In Section 2, we extend the rectifiability theorem by extending Blow-up Theorem ([11], Theorem 3.1). In Section 3, we prove an interesting property of Carnot groups of type \star . In Section 4, we show examples of Carnot groups for which Blow-up Theorem does not hold, besides the well known Engel group (see [11], Example 3.2).

Now we recall some preliminary definition and results on Carnot groups. For more details, one can see [4] and [11].

Definition 1.1. A Carnot group \mathbb{G} of step k is a nilpotent connected and simply connected Lie group, whose Lie algebra \mathfrak{g} admit a step k stratification, i.e. there exist linear subspaces V_1, \dots, V_k such that

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \quad (1.1)$$

where $[V_1, V_i]$ is the subspace of \mathfrak{g} generated by commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

We set $m_i := \dim(V_i)$ for $i = 1, \dots, k$.

If we think of \mathfrak{g} as the tangent space at the identity element of \mathbb{G} (denoted with 0), and consider a basis of \mathfrak{g} (e_1, \dots, e_n) adapted to its stratification, we can define the canonical vector fields as the left invariant vector fields X_1, \dots, X_n such that $X_i(0) = e_i$.

For (1.1), vector fields X_1, \dots, X_{m_1} generate all the left invariant vector fields, hence we will call them generating vector fields of the group.

Since Carnot groups are nilpotent, connected and simply connected, the exponential map is a diffeomorphism from \mathfrak{g} to \mathbb{G} , i.e. every $p \in \mathbb{G}$ can be uniquely written in the form

$$p = \exp(p_1 X_1 + \cdots + p_n X_n). \quad (1.2)$$

By using this exponential coordinates, we can identify p with the n -tuple

$$(p_1, \dots, p_n) \in \mathbb{R}^n$$

and identify \mathbb{G} with (\mathbb{R}^n, \cdot) where the explicit expression of the group operation \cdot is determined by Campbell-Hausdorff formula (see [8]). More precisely, \mathbb{G} is isomorphic to the Lie group (\mathbb{R}^n, \cdot) .

A Carnot group is characterized by a horizontal bundle, which is the sub-bundle of $T\mathbb{G}$ that is spanned by vector fields X_1, \dots, X_{m_1} and it is denoted with $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span} \{X_1(x), \dots, X_{m_1}(x)\}, \quad x \in \mathbb{G}.$$

Moreover, every fiber of $H\mathbb{G}$ is endowed with an inner product $\langle \cdot, \cdot \rangle_x$ and a norm $|\cdot|_x$; precisely, if $v = \sum_{i=1}^{m_1} v_i X_i(x) = (v_1, \dots, v_{m_1})$ and $w = \sum_{i=1}^{m_1} w_i X_i(x) = (w_1, \dots, w_{m_1})$ are in $H\mathbb{G}_x$, then

$$\langle v, w \rangle_x := \sum_{j=1}^{m_1} v_j w_j \quad \text{and} \quad |v|_x^2 := \langle v, v \rangle_x.$$

Definition 1.2. For each $x \in \mathbb{G}$, we define the left translation by x as

$$\begin{aligned} \tau_x : \mathbb{G} &\rightarrow \mathbb{G} \\ z &\mapsto x \cdot z \end{aligned}$$

and, for each $\lambda > 0$, we define the dilation $\delta_\lambda : \mathbb{G} \rightarrow \mathbb{G}$ as

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n), \quad (1.3)$$

where $\alpha_i \in \mathbb{N}$ is the homogeneity of variable x_i in \mathbb{G} and it is defined as

$$\alpha_i = j \quad \text{when} \quad 1 + \sum_{k=1}^{j-1} m_k \leq i \leq \sum_{k=1}^j m_k. \quad (1.4)$$

Definition 1.3. An absolutely continuous curve $\gamma : [0, T] \rightarrow \mathbb{G}$ is a sub-unit curve with respect to X_1, \dots, X_{m_1} if it is a horizontal curve, i.e. there exist measurable real functions $c_1(s), \dots, c_{m_1}(s)$, $s \in [0, T]$ such that

$$\dot{\gamma}(s) = \sum_{j=1}^{m_1} c_j(s) X_j(\gamma(s)), \quad \text{for a.e. } s \in [0, T],$$

and if

$$\sum_j c_j^2 \leq 1.$$

Definition 1.4. If $p, q \in \mathbb{G}$, we define their Carnot-Carathéodory distance as

$$d_c(p, q) := \inf \{T > 0 : \text{there exists a sub-unit curve } \gamma \text{ with } \gamma(0) = p, \gamma(T) = q\}.$$

The set of sub-unit curves joining p and q is not empty, since Chow's Theorem (see [20], Theorem 1.6.2); furthermore d_c is a distance on \mathbb{G} that induces the Euclidean topology (see [21]). We denote with $U_c(p, r)$ and $B_c(p, r)$ respectively the open and closed balls associated with d_c .

Definition 1.5. We also define another distance equivalent to the previous one.

$$d_\infty(x, y) = d_\infty(y^{-1} \cdot x, 0),$$

where, if $p = (\tilde{p}_1, \dots, \tilde{p}_k) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k} = \mathbb{R}^n$, then

$$d_\infty(p, 0) = \max\{\varepsilon_j \|\tilde{p}_j\|_{\mathbb{R}^{m_j}}^{1/j}, j = 1, \dots, k\}. \quad (1.5)$$

Here $\varepsilon_1 = 1$, and $\varepsilon_2, \dots, \varepsilon_k \in (0, 1)$ are suitable positive constants depending on the group structure (see [11], Theorem 5.1).

Definition 1.6. The integer

$$Q = \sum_{j=1}^n \alpha_j = \sum_{i=1}^k i \dim V_i \quad (1.6)$$

is the homogeneous dimension of \mathbb{G} . We stress that it is also the Hausdorff dimension of \mathbb{R}^n with respect to d_c (see [19]).

Proposition 1.7. *The n -dimensional Lebesgue measure \mathcal{L}^n is the Haar measure of the group \mathbb{G} (see [23]). Therefore if $E \subset \mathbb{R}^n$ is measurable, then $\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E)$ for every $x \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E)$. We note that*

$$\mathcal{L}^n(U_c(p, r)) = r^Q \mathcal{L}^n(U_c(p, 1)) = r^Q \mathcal{L}^n(U_c(0, 1)). \quad (1.7)$$

In this article, all the spaces $L^p(\mathbb{G})$ are defined with respect to the Haar measure of the group.

Definition 1.8. Let Ω be an open set in \mathbb{G} , then $f : \Omega \rightarrow \mathbb{R}$ is Pansu differentiable (see [22] and [16]) in x_0 if there exists a \mathbb{G} -linear map L such that

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - L(x_0^{-1} \cdot x)}{d_c(x, x_0)} = 0.$$

An equivalent definition is the following one: there exists a group homomorphism L from \mathbb{G} to $(\mathbb{R}, +)$ such that

$$\lim_{\lambda \rightarrow 0+} \frac{f(\tau_{x_0}(\delta_\lambda v)) - f(x_0)}{\lambda} = L(v)$$

uniformly with respect to v belonging to compact sets in \mathbb{G} .

In particular, L is unique and we write $L = d_{\mathbb{G}}f(x_0)$. We remark that Pansu differential depends only on \mathbb{G} and not on a particular choice of the canonical vector fields.

Definition 1.9. If Ω is an open set in \mathbb{G} , we denote with $\mathbf{C}_{\mathbb{G}}^1(\Omega)$ the set of continuous real functions in Ω such that $d_{\mathbb{G}}f : \Omega \rightarrow \mathcal{L}_{\mathbb{G}}$ is continuous in Ω . Furthermore, we denote with $\mathbf{C}_{\mathbb{G}}^1(\Omega, H\mathbb{G})$ the set of all sections ϕ of $H\mathbb{G}$ whose canonical coordinates $\phi_j \in \mathbf{C}_{\mathbb{G}}^1(\Omega)$ for $j = 1, \dots, m_1$.

Definition 1.10. If we fix a generating family of vector fields X_1, \dots, X_{m_1} , we can define, for any regular function $f : \mathbb{G} \rightarrow \mathbb{R}$, the horizontal gradient of f as the horizontal section

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m_1} (X_i f) X_i.$$

whose coordinates are $(X_1 f, \dots, X_{m_1} f)$.

Definition 1.11. If $\phi = (\phi_1, \dots, \phi_{m_1})$ is an horizontal section such that $X_j \phi_j \in L^1_{\text{loc}}(\mathbb{G})$ for $j = 1, \dots, m_1$, we define the horizontal divergence of ϕ as the real valued function

$$\text{div}_{\mathbb{G}}(\phi) := \sum_{j=1}^{m_1} X_j \phi_j.$$

If $\Omega \subset \mathbb{G}$ is open, the space of compactly supported smooth sections of $H\mathbb{G}$ is denoted by $\mathbf{C}_0^\infty(\Omega, H\mathbb{G})$. If $k \in \mathbb{N}$, $\mathbf{C}_0^k(\Omega, H\mathbb{G})$ is defined similarly.

Definition 1.12. We say that a function $f \in L^1(\Omega)$ is of bounded variation in Ω if

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup \left\{ \int_{\Omega} f(x) \text{div}_{\mathbb{G}} \phi(x) dx : \phi \in \mathbf{C}_0^1(\Omega, H\mathbb{G}), |\phi(x)|_x \leq 1 \right\} < \infty.$$

We denote the set of all these functions with $BV_{\mathbb{G}}(\Omega)$, whereas we denote with $BV_{\mathbb{G}, \text{loc}}(\Omega)$ the set of functions belonging to $BV_{\mathbb{G}}(\mathcal{U})$ for every open set $\mathcal{U} \subset \subset \Omega$.

Theorem 1.13. *If $f \in BV_{\mathbb{G}, \text{loc}}(\Omega)$ then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on Ω . Moreover, there exists a $\|\nabla_{\mathbb{G}} f\|$ -measurable horizontal section $\sigma_f : \Omega \rightarrow H\mathbb{G}$ such that $|\sigma_f(x)|_x = 1$ for $\|\nabla_{\mathbb{G}} f\|$ -a.e. $x \in \Omega$, and*

$$\int_{\Omega} f(x) \text{div}_{\mathbb{G}} \phi(x) dx = \int_{\Omega} \langle \phi, \sigma_f \rangle d\|\nabla_{\mathbb{G}} f\|,$$

for every $\phi \in \mathbf{C}_0^1(\Omega, H\mathbb{G})$. Thus, the notion of gradient $\nabla_{\mathbb{G}}$ can be extended from regular functions to functions $f \in BV_{\mathbb{G}}$ defining $\nabla_{\mathbb{G}} f$ as the vector valued measure

$$\nabla_{\mathbb{G}} f := -\sigma_f \llcorner \|\nabla_{\mathbb{G}} f\| = (-(\sigma_f)_1 \llcorner \|\nabla_{\mathbb{G}} f\|, \dots, -(\sigma_f)_{m_1} \llcorner \|\nabla_{\mathbb{G}} f\|),$$

where $(\sigma_f)_j$ are the components of σ_f with respect to the moving base X_j .

For more details and proofs, see [14] and [12].

Definition 1.14. A measurable set $E \subset \mathbb{G}$ is of locally finite perimeter in Ω if the characteristic function $\mathbf{1}_E \in BV_{\mathbb{G}, \text{loc}}(\Omega)$. In this case we call perimeter of E in Ω the measure

$$|\partial E|_{\mathbb{G}} := \|\nabla_{\mathbb{G}} \mathbf{1}_E\| \tag{1.8}$$

and we call generalized inward \mathbb{G} -normal to ∂E in Ω the vector

$$\nu_E(x) := -\sigma_{\mathbf{1}_E}(x). \tag{1.9}$$

We say that E is a \mathbb{G} -Caccioppoli set if it is of locally finite perimeter in \mathbb{G} .

Similarly as in the Euclidean setting, given $E \subset \mathbb{G}$, we define the essential boundary $\partial_{*, \mathbb{G}} E$ and the reduced boundary $\partial_{\mathbb{G}}^* E$.

Definition 1.15. Let $E \subset \mathbb{G}$ be a measurable set, we say that $x \in \partial_{*,\mathbb{G}}E$ if

$$\limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^n(E \cap U_c(x, r))}{\mathcal{L}^n(U_c(x, r))} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^n(E^c \cap U_c(x, r))}{\mathcal{L}^n(U_c(x, r))} > 0.$$

Definition 1.16. Let E be a \mathbb{G} -Caccioppoli set; we say that $x \in \partial_{\mathbb{G}}^*E$ if

$$|\partial E|_{\mathbb{G}}(U_c(x, r)) > 0 \quad \text{for any } r > 0; \quad (i)$$

$$\text{there exists } \lim_{r \rightarrow 0} \int_{U_c(x, r)} \nu_E d|\partial E|_{\mathbb{G}}; \quad (ii)$$

$$\left\| \lim_{r \rightarrow 0} \int_{U_c(x, r)} \nu_E d|\partial E|_{\mathbb{G}} \right\|_{\mathbb{R}^{m_1}} = 1. \quad (iii)$$

Lemma 1.17 (Differentiation Lemma). Assume E is a \mathbb{G} -Caccioppoli set, then

$$\lim_{r \rightarrow 0} \int_{U_c(x, r)} \nu_E d|\partial E|_{\mathbb{G}} = \nu_E(x), \quad \text{for } |\partial E|_{\mathbb{G}}\text{-a.e. } x,$$

hence $|\partial E|_{\mathbb{G}}$ is concentrated on the reduced boundary $\partial_{\mathbb{G}}^*E$.

Remark 1.18. Thanks to Lemma 1.17, we can redefine ν_E in a $|\partial E|_{\mathbb{G}}$ -negligible set, by assuming that ν_E is the limit of the averages at all point $x \in \partial_{\mathbb{G}}^*E$.

Definition 1.19. $S \subset \mathbb{G}$ is a \mathbb{G} -regular hypersurface if for every $x \in S$ there exist a neighborhood \mathcal{U} of x and a function $f \in \mathbf{C}_{\mathbb{G}}^1(\mathcal{U})$ such that

$$S \cap \mathcal{U} = \{y \in \mathcal{U} : f(y) = 0\}; \quad (i)$$

$$\nabla_{\mathbb{G}} f(y) \neq 0 \quad \text{for } y \in \mathcal{U}. \quad (ii)$$

Definition 1.20. $\Gamma \subset \mathbb{G}$ is said to be $(Q-1)$ -dimensional \mathbb{G} -rectifiable if there exists a sequence of \mathbb{G} -regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ such that

$$\mathcal{H}_c^{Q-1} \left(\Gamma \setminus \bigcup_{j \in \mathbb{N}} S_j \right) = 0, \quad (1.10)$$

where \mathcal{H}_c^{Q-1} is the $(Q-1)$ -dimensional Hausdorff measure related to the distance d_c .

Definition 1.21. If $S = \{x : f(x) = 0\} \subset \mathbb{G}$ is a \mathbb{G} -regular hypersurface, the tangent group to S at x is the proper subgroup of \mathbb{G} defined as

$$T_{\mathbb{G}}^g S(x) := \{v \in \mathbb{G} : \langle \nabla_{\mathbb{G}} f(x), \pi_x v \rangle_x = 0\},$$

where $\pi_x v := \sum_{j=1}^{m_1} v_j X_j(x)$.

We can also define the tangent plane to S at x as

$$T_{\mathbb{G}} S(x) := x \cdot T_{\mathbb{G}}^g S(x).$$

This definition is good; in fact the tangent plane does not depend on the particular function f defining the surface S because of point (iii) of Theorem 2.1 (Implicit Function Theorem) in [13] that yields

$$T_{\mathbb{G}}^g S(x) = \{v \in \mathbb{G} : \langle \nu_E(x), \pi_x v \rangle_x = 0\}$$

where ν_E is the generalized inward unit normal defined in (1.9).

We stress that the notion of \mathbb{G} -regular hypersurfaces is different from the one of Euclidean \mathbf{C}^1 -hypersurfaces in \mathbb{R}^n . In particular, in Corollary 1.25 we will consider

Euclidean \mathbf{C}^1 -surfaces, which can have characteristic points, i.e. points $p \in S$ where the Euclidean tangent plane $T_p S$ contains the horizontal fiber $H\mathbb{G}_p$. If S is an Euclidean \mathbf{C}^1 -hypersurface in \mathbb{G} , we denote with $\mathcal{C}(S)$ the set of its characteristic points. The tangent group does not exist in these points; however, there is an important result about them proved in [17]: in any Carnot group it holds that, if S is a \mathbf{C}^1 -hypersurface, $\mathcal{H}_c^{Q-1}(\mathcal{C}(S)) = 0$.

In [11], the rectifiability theorem is proved for step 2 Carnot groups and Blow-up Theorem is the main key of the proof and also the reason of the restriction to step 2. In fact, there is a counterexample regarding a particular step 3 Carnot group, i.e. the Engel group, for which Blow-up Theorem does not hold (see [11], Example 3.2). The problem of rectifiability in general Carnot groups remains an open question. Here we recall the Blow-up Theorem.

Let \mathbb{G} be a Carnot group. For any set $E \subset \mathbb{G}$, $x_0 \in \mathbb{G}$ and $r > 0$ we define the sets

$$E_{r,x_0} := \{x : x_0 \cdot \delta_r(x) \in E\} = \delta_{\frac{1}{r}} \tau_{x_0^{-1}} E.$$

If $v \in H\mathbb{G}_{x_0}$ we define the halfspaces $S_{\mathbb{G}}^+(v)$ and $S_{\mathbb{G}}^-(v)$ as

$$\begin{aligned} S_{\mathbb{G}}^+(v) &:= \{x : \langle \pi_{x_0} x, v \rangle_{x_0} \geq 0\} \\ S_{\mathbb{G}}^-(v) &:= \{x : \langle \pi_{x_0} x, v \rangle_{x_0} \leq 0\}. \end{aligned} \quad (1.11)$$

The common topological boundary $T_{\mathbb{G}}^g(v)$ of $S_{\mathbb{G}}^+(v)$ and of $S_{\mathbb{G}}^-(v)$ is the subgroup of \mathbb{G}

$$T_{\mathbb{G}}^g(v) := \{x : \langle \pi_{x_0} x, v \rangle_{x_0} = 0\}.$$

Moreover, we shall denote with \mathcal{H}^{n-1} the $(n-1)$ -dimensional Hausdorff measure related to the Euclidean distance in $\mathbb{R}^n \simeq \mathbb{G}$, with \mathcal{S}_c^{Q-1} the $(Q-1)$ -dimensional spherical Hausdorff measure related to the distance d_c in \mathbb{G} , and with $\mathcal{S}_{\infty}^{Q-1}$ the $(Q-1)$ -dimensional spherical Hausdorff measure related to the distance d_{∞} in \mathbb{G} .

Theorem 1.22 (Blow-up). *If E is a \mathbb{G} -Caccioppoli set in a step 2 Carnot group \mathbb{G} , $x_0 \in \partial_{\mathbb{G}}^* E$ and $\nu_E(x_0) \in H\mathbb{G}_{x_0}$ is the inward normal then*

$$\lim_{r \rightarrow 0} \mathbf{1}_{E_{r,x_0}} = \mathbf{1}_{S_{\mathbb{G}}^+(\nu_E(x_0))} \quad \text{in } L_{\text{loc}}^1(\mathbb{G}) \quad (1.12)$$

and for all $R > 0$

$$\lim_{r \rightarrow 0} |\partial E_{r,x_0}|_{\mathbb{G}}(U_c(0, R)) = |\partial S_{\mathbb{G}}^+(\nu_E(x_0))|_{\mathbb{G}}(U_c(0, R)). \quad (1.13)$$

and

$$|\partial S_{\mathbb{G}}^+(\nu_E(x_0))|_{\mathbb{G}}(U_c(0, R)) = \mathcal{H}^{n-1}(T_{\mathbb{G}}^g(\nu_E(0)) \cap U_c(0, R)).$$

Theorem 1.23 (Rectifiability Theorem). *If $E \subseteq \mathbb{G}$ is a \mathbb{G} -Caccioppoli set in a step 2 Carnot group \mathbb{G} , then*

$$\partial_{\mathbb{G}}^* E \text{ is } (Q-1)\text{-dimensional } \mathbb{G}\text{-rectifiable,} \quad (i)$$

that is $\partial_{\mathbb{G}}^* E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $\mathcal{H}_c^{Q-1}(N) = 0$ and K_h is a compact subset of a \mathbb{G} -regular hypersurface S_h ;

$$\nu_E(p) \text{ is the } \mathbb{G}\text{-normal to } S_h \text{ at } p, \text{ for all } p \in K_h; \quad (ii)$$

$$|\partial E|_{\mathbb{G}} = \theta_c \mathcal{S}_c^{Q-1} \llcorner \partial_{\mathbb{G}}^* E, \quad (iii)$$

where

$$\theta_c(x) = \frac{1}{\omega_{Q-1}} \mathcal{H}^{n-1} (\partial S_{\mathbb{G}}^+(\nu_E(x)) \cap U_c(0, 1)).$$

Here ω_k is the k -dimensional measure of the k -dimensional ball in \mathbb{R}^k . If we replace the \mathcal{S}_c -measure by the \mathcal{S}_∞ -measure, the corresponding density θ_∞ is a constant. More precisely

$$|\partial E|_{\mathbb{G}} = \theta_\infty \mathcal{S}_\infty^{Q-1} \llcorner \partial_{\mathbb{G}}^* E, \quad (iv)$$

where

$$\theta_\infty = \frac{\omega_{m_1-1} \omega_{m_2} \varepsilon_2^{m_2}}{\omega_{Q-1}} = \frac{1}{\omega_{Q-1}} \mathcal{H}^{n-1} (\partial S_{\mathbb{G}}^+(\nu_E(0)) \cap U_\infty(0, 1)).$$

Here ε_2 is a constant that appears in (1.5).

A consequence of Theorem 1.23 is the following divergence theorem.

Theorem 1.24 (Divergence Theorem). *Let E be a \mathbb{G} -Caccioppoli set in a step 2 Carnot group \mathbb{G} , then*

$$|\partial E|_{\mathbb{G}} = \theta_\infty \mathcal{S}_\infty^{Q-1} \llcorner \partial_{*,\mathbb{G}} E, \quad (i)$$

and the following version of the divergence theorem holds

$$-\int_E \operatorname{div}_{\mathbb{G}} \phi \, d\mathcal{L}^n = \theta_\infty \int_{\partial_{*,\mathbb{G}} E} \langle \nu_E, \phi \rangle \, d\mathcal{S}_\infty^{Q-1}, \quad \forall \phi \in \mathbf{C}_0^1(\mathbb{G}, H\mathbb{G}). \quad (ii)$$

In case the boundary of E is of class \mathbf{C}^1 , a sharper result holds.

Corollary 1.25. *If \mathbb{G} is a Carnot group of step 2 and a measurable set $E \subset \mathbb{G}$ has boundary of class \mathbf{C}^1 (and hence E is a \mathbb{G} -Caccioppoli set), then*

$$|\partial E|_{\mathbb{G}} = \theta_\infty \mathcal{S}_\infty^{Q-1} \llcorner \partial E = \left(\sum_{j=1}^{m_1} \langle X_j, n_E \rangle_{\mathbb{R}^n}^2 \right)^{1/2} \mathcal{H}^{n-1} \llcorner \partial E. \quad (i)$$

where n_E denotes the Euclidean outward normal to ∂E . Again a version of the divergence theorem holds

$$-\int_E \operatorname{div}_{\mathbb{G}} \phi \, dx = \theta_\infty \int_{\partial E} \langle \nu_E, \phi \rangle \, d\mathcal{S}_\infty^{Q-1} \quad \forall \phi \in \mathbf{C}_0^1(\mathbb{G}, H\mathbb{G}). \quad (ii)$$

In order to prove this corollary, in [11], it is shown that (in a step 2 Carnot group) if S is a \mathbf{C}^1 -hypersurface, then $\mathcal{H}_c^{Q-1}(\mathcal{C}(S)) = 0$ (see [11], Theorem 4.8). Anyway, Magnani extended this result to any Carnot group in [17]. Since non-characteristic points of a boundary ∂E of class \mathbf{C}^1 belongs to the reduced boundary, we conclude that $\mathcal{S}_\infty^{Q-1}(\partial E \setminus \partial_{\mathbb{G}}^* E) = 0$.

2. EXTENSION OF RECTIFIABILITY THEOREM TO CARNOT GROUPS OF TYPE \star

In this section we use the notions of free and filiform Carnot groups. One can see [4] for details.

From now on, we set $m := m_1$.

Definition 2.1. We say that a Carnot group \mathbb{G} is of type \star if its stratified Lie algebra $\mathfrak{g} = V_1 \oplus \dots \oplus V_k$ has the following property: there exists a basis (X_1, \dots, X_m) of V_1 such that

$$[X_j, [X_j, X_i]] = 0 \text{ for } i, j = 1, \dots, m \quad (2.1)$$

In this case we also say that \mathfrak{g} is of type \star .

Remark 2.2. The previous definition is well posed, i.e. the property is invariant under Lie algebra isomorphisms that respect stratification.

It is clear that every step 2 Carnot group is of type \star , whereas free Carnot groups of step greater than 2 are not of type \star . Moreover, if a Carnot group of step greater than 2 is of type \star , then the dimension of its first layer is at least 3; hence filiform groups of step greater than 2 (and in particular Engel group) are not of type \star .

Example 2.3. The Lie group \mathbb{G}_m of unit upper triangular $(m+1) \times (m+1)$ matrices is a Carnot group of type \star , for any $m \in \mathbb{N}$. However, in order to avoid trivial cases, it is possible to assume $m > 2$. This group is the nilpotent group that comes from the Iwasawa decomposition of $GL_{m+1}(\mathbb{R})$.

Now let us prove that \mathbb{G}_m is of type \star for $m > 2$. The Lie algebra \mathfrak{g}_m of \mathbb{G}_m is isomorphic to the one of strictly upper triangular $(m+1) \times (m+1)$ matrices (see [15], Part I, Chapter 2, Section 5.7, Example 1). If $E_{i,j}$ is the matrix with 1 in the (i,j) -th entry and 0 elsewhere, it is easy to see that a basis of \mathfrak{g}_m is formed by the single-entry matrices $E_{k,k+l}$ for $l = 1, \dots, m$ and $k = 1, \dots, m+1-l$, and $\dim \mathfrak{g}_m = \frac{m(m+1)}{2}$. The choice of using the particular parameters k and l will soon be explained.

The following formula, which can be proven by direct computation of the commutators of single-entry matrices, gives the expression of Lie brackets in \mathfrak{g}_m

$$\begin{aligned} \text{If } k_1 < k_2, \text{ then } [E_{k_1, k_1+l_1}, E_{k_2, k_2+l_2}] &= \begin{cases} E_{k_1, k_1+l_1+l_2} & \text{if } k_1 + l_1 = k_2 \\ 0 & \text{if } k_1 + l_1 \neq k_2 \end{cases} \\ \text{If } k_1 > k_2, \text{ then } [E_{k_1, k_1+l_1}, E_{k_2, k_2+l_2}] &= -[E_{k_2, k_2+l_2}, E_{k_1, k_1+l_1}] \\ \text{If } k_1 = k_2, \text{ then } [E_{k_1, k_1+l_1}, E_{k_2, k_2+l_2}] &= 0 \end{aligned} \quad (2.2)$$

From (2.2), it is easy to see that $E_{k, k+1}$ (for $k = 1, \dots, m$) are generators of \mathfrak{g}_m . Moreover, $\mathfrak{g}_m = V_1 \oplus \dots \oplus V_m$ with

$$V_l = \text{span}\{E_{k, k+l} \mid k = 1, \dots, m+1-l\}$$

for $l = 1, \dots, m$. This explains the use of the parameters k and l . Moreover, we observe that m is the dimension of V_1 and the step of the stratification.

Now we can finally prove that \mathbb{G}_m is of type \star . We set $X_k := E_{k, k+1}$ for $k = 1, \dots, m$. From (2.2) we obtain that $E_{k, k+2} = [X_k, X_{k+1}]$ for $k = 1, \dots, m-1$ and the other independent commutators of length 2 are zero, whereas $E_{k, k+3} = [[X_k, X_{k+1}], X_{k+2}] = [X_k, [X_{k+1}, X_{k+2}]]$ for $k = 1, \dots, m-2$ and the other independent commutators of length 3 are zero. Hence (2.1) holds.

The identity (2.2) allows to explicitly write an adapted basis of \mathfrak{g}_m and the expression of Lie brackets for any $m \in \mathbb{N}$, but we also want to remark the trivial case obtained when $m = 2$, that is the Lie algebra of the three-dimensional Heisenberg group.

Another example of stratified algebra of type \star (besides \mathfrak{g}_m) is obtained from

$$\mathfrak{g}_3 = \text{span}\{X_1, X_2, X_3\} \oplus \text{span}\{[X_1, X_2], [X_2, X_3]\} \oplus \text{span}\{[[X_1, X_2], X_3]\},$$

where $[X_1, X_3] = 0$, $[[X_1, X_2], X_3] = [X_1, [X_2, X_3]]$ and the other commutators of length 3 are zero, by setting $[X_1, X_3] = b[X_2, X_3]$ with $b \neq 0$. This can be done, since Jacobi identity

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0$$

is still verified. The only non-zero commutators of length 3 are

$$[[X_1, X_2], X_3] = [X_1, [X_2, X_3]] = [[X_3, X_2], X_1] \text{ and } [[X_3, X_1], X_2] = b[[X_1, X_2], X_3]$$

By changing the basis of the first layer in the following way

$$\begin{cases} \tilde{X}_1 = X_1 - bX_2 \\ \tilde{X}_2 = X_2 \\ \tilde{X}_3 = X_3 \end{cases}$$

it is easy to see that \mathfrak{g} is of type \star .

Definition 2.4. An ideal of a graded Lie algebra is said to be homogeneous if can be generated by homogeneous elements.

Obviously, the quotient of a stratified Lie algebra by an homogeneous ideal is still a stratified Lie algebra.

Remark 2.5. A stratified Lie algebra of type \star can contain filiform stratified subalgebras of step greater than 2.

For instance, let us consider the free Lie algebra of step 3 with 3 generators. Now we quotient it by the homogeneous ideal generated by $[X_j, [X_j, X_i]]$ with $i, j = 1, 2, 3$. Obviously the obtained stratified Lie algebra is of type \star . If we consider its stratified subalgebra $\text{Lie}\{X_1 + X_2, X_3\}$, we can verify it is filiform of step 3.

$$\begin{aligned} [X_1 + X_2, X_3] &= [X_1, X_3] + [X_2, X_3] \neq 0 \\ [X_1 + X_2, [X_1 + X_2, X_3]] &= [X_2, [X_1, X_3]] + [X_1, [X_2, X_3]] \neq 0 \\ [X_3, [X_1 + X_2, X_3]] &= 0. \end{aligned}$$

Thus, type \star property is not inherited by stratified subalgebras, but is obviously inherited by stratified quotient algebras.

Remark 2.6. In a stratified algebra, $\dim V_3 \leq \frac{1}{3}(m+1)m(m-1)$ and $\dim V_3 = \frac{1}{3}(m+1)m(m-1)$ in free stratified algebras of step greater than 2. In fact, there are $m(m-1)$ commutators of form $[X_j, [X_j, X_i]]$ with $j \neq i$, $\binom{m}{3}$ commutators of form $[X_i, [X_j, X_k]]$ with $i < j < k$ and $\binom{m}{3}$ commutators of form $[X_k, [X_j, X_i]]$ with $i < j < k$, which span V_3 and are linearly independent, if we consider only the relations of antisymmetry and Jacobi identities. If we sum the number of these commutators, we get

$$m(m-1) + 2\binom{m}{3} = \frac{1}{3}(m+1)m(m-1).$$

On the contrary, in a stratified Lie algebra of type \star , $\dim V_3 \leq \frac{1}{3}m(m-1)(m-2)$.

For a dimension reason we can say again that free stratified algebras of step greater than 2 are not of type \star and that stratified algebras of type \star with $m = 2$ cannot be of step greater than 2, hence filiform algebras of step greater than 2 are not of type \star .

The main result of this paper reads as follows.

Theorem 2.7 (Blow-up extended). *Theorem 1.22 still holds in Carnot groups of type \star .*

Proof. The proof is the same as the one in [11], with the exception of Lemma 3.2 therein, that is extended here to Carnot groups of type \star (see Proposition 2.9). In fact, we would need to extend to our setting also Theorem 4.8 of [11] on the Hausdorff measure of the characteristic set of regular Euclidean hypersurface, but nowadays this result is known to hold for any Carnot group, thanks to [17].

Thus, from now on we restrict ourselves to provide the modified part of the proof. Preliminarily, we need Lemma 2.8 below, which shows an important property of stratified Lie algebras of type \star .

Lemma 2.8. *Let $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ be a stratified Lie algebra of type \star . Then for every basis (Y_1, \dots, Y_m) of V_1*

$$[Y_1, [Y_1, Y_p]] = \sum_{j>1} \alpha_{pij} [Y_j, [Y_j, Y_i]] + \sum_{k \neq j, k \neq i} \beta_{pijk} [Y_k, [Y_j, Y_i]]$$

holds for $p = 2, \dots, m$ (with $\alpha_{pij}, \beta_{pijk} \in \mathbb{R}$).

We stress that the first sum contains commutators “with repeated indices different from 1”, whereas the second one contains commutators “without repeated indices”. Roughly speaking, the previous lemma states that a commutator where an index i is repeated (for instance $i = 1$) can be written as a linear combination of the remaining commutators excluding those where the index i is repeated.

Proof. Let (Y_1, Y_2, \dots, Y_m) be any basis of V_1 . Obviously

$$V_3 = \text{span}\{[Y_k, [Y_j, Y_i]] \mid i, j, k = 1, \dots, m\}.$$

Now let (X_1, X_2, \dots, X_m) be a basis of V_1 that respects (2.1). The relation between the two basis is

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} = A \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix} \quad (2.3)$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

invertible real $m \times m$ matrix. Hence there exists $i \in \{1, \dots, m\}$ such that $a_{i1} \neq 0$. After reordering the basis (X_1, X_2, \dots, X_m) , we can assume $a_{11} \neq 0$. Now we recall the relations

$$[X_1, [X_1, X_h]] = 0 \quad \text{for } h = 2, \dots, m.$$

Replacing (2.3) in these identities, we get

$$\sum_{i,j,k} a_{1k} a_{1j} a_{hi} [Y_k, [Y_j, Y_i]] = 0 \quad \text{for } h = 2, \dots, m.$$

Keeping in mind the antisymmetry of brackets, we obtain

$$\sum_{i=2}^m (a_{11} a_{11} a_{hi} - a_{11} a_{h1} a_{1i}) [Y_1, [Y_1, Y_i]] + \sum_k \alpha_{hk} Z_k = 0 \quad \text{for } h = 2, \dots, m$$

where Z_k are all the remaining commutators of length 3, i.e. commutators that are not of the form $[Y_1, [Y_i, Y_1]]$ with $i = 2, \dots, m$ which are all incorporated in the first sum. Now we move all terms containing Z_k to the right-hand side and we get

$$\sum_{i=2}^m (a_{11}a_{hi} - a_{h1}a_{1i})[Y_1, [Y_1, Y_i]] = -\frac{1}{a_{11}} \sum_k \alpha_{hk} Z_k \text{ for } h = 2, \dots, m.$$

If we denote by U_ℓ the ℓ -th component of a vector $U \in V_3$, $\ell = 1, \dots, \dim V_3$ with respect to an arbitrarily fixed basis, then the components $[Y_1, [Y_1, Y_i]]_\ell$ of $[Y_1, [Y_1, Y_i]]$ are solutions of a $(m-1) \times (m-1)$ linear system with coefficient matrix

$$M = (m_{ij})_{i,j=1,\dots,m-1} \quad \text{where } m_{ij} = a_{1,1}a_{i+1,j+1} - a_{i+1,1}a_{1,j+1}$$

We stress that m_{ij} are the second order minors of A that contains a_{11} . M is invertible since

$$\det M = (a_{11})^{m-2} \det A \neq 0.$$

This computation comes from the so-called *Chio's pivotal condensation* (see, e.g., [7], Theorem 3.6.1). Therefore, we obtain eventually

$$\begin{pmatrix} ([Y_1, [Y_1, Y_2]])_\ell \\ \dots \\ ([Y_1, [Y_1, Y_m]])_\ell \end{pmatrix} = -\frac{1}{a_{11}} \sum_k (Z_k)_\ell M^{-1} \begin{pmatrix} \alpha_{2k} \\ \dots \\ \alpha_{mk} \end{pmatrix} \quad \text{for } \ell = 1, \dots, \dim V_3$$

Then the thesis follows straightforwardly.

Proposition 2.9. *Let \mathbb{G} be a Carnot group of type \star and let Y_1, \dots, Y_m be left invariant orthonormal sections of $H\mathbb{G}$. Suppose $g \in L^1_{\text{loc}}(\mathbb{G})$ satisfies*

$$Y_1 g \geq 0 \quad \text{and} \quad Y_j g = 0 \quad \text{for } j = 2, \dots, m. \quad (2.4)$$

Then the level lines of g are “vertical hyperplanes orthogonal to Y_1 ” that is sets that are group translations of

$$S(Y_1) := \{p \mid \langle \pi_0 p, Y_1(0) \rangle = 0\}.$$

Proof. We can assume that g is smooth since the general case can be proven approximating g by the group convolution (see [9]). We observe that Y_1, \dots, Y_m form a basis for V_1 . If (X_1, \dots, X_m) is a basis for V_1 satisfying (2.1), then

$$Y_1 = \sum a_{1i} X_i, \dots, Y_m = \sum a_{mi} X_i. \quad (2.5)$$

We denote with A the coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \quad \det A \neq 0 \quad (2.6)$$

Now we notice that (2.5) yields

$$\begin{pmatrix} Y_1 g(x) \\ \dots \\ Y_m g(x) \end{pmatrix} = A \begin{pmatrix} X_1 g(x) \\ \dots \\ X_m g(x) \end{pmatrix} \quad \text{for every } x \in \mathbb{G}$$

From (2.6) A is invertible, hence the inverse image of $\{(t, 0, \dots, 0) \in \mathbb{R}^m \mid t \in \mathbb{R}\}$ under A is a line of \mathbb{R}^m , i.e. there exists $k \in \mathbb{R}^m$, $k \neq 0$ such that

$$A^{-1}\{(t, 0, \dots, 0) \in \mathbb{R}^m \mid t \in \mathbb{R}\} = \{\lambda k \mid \lambda \in \mathbb{R}\} \quad (2.7)$$

Since $k = (k_1, \dots, k_m) \neq 0$, at least one of its components is not zero: for instance we can assume $k_m \neq 0$ and then, without loss of generality, $k_m = 1$. We stress that k depends only on A , so that the vector k is independent of $x \in \mathbb{G}$.

If we indicate with A_j the j -th row of A , we have

$$\langle A_1, k \rangle \neq 0. \quad (2.8)$$

Indeed by the definition of k , $\langle A_j, k \rangle = 0$ for $j = 2, \dots, m$ and $\langle A_1, k \rangle \neq 0$ by the invertibility of A .

Remember now that, by hypothesis, $Y_1 g \geq 0$, $Y_2 g = 0$, \dots , $Y_m g = 0$. Therefore, if $x \in \mathbb{G}$, then by (2.7), $X_i g(x) = \lambda(x) k_i$ for $i = 1, \dots, m$. In particular, taking $i = m$, we have $\lambda(x) = X_m g(x)$, which leads to $X_i g(x) = k_i X_m g(x)$ for $i = 1, \dots, m$.

Now $Y_1 g(x) = \langle A_1, k \rangle X_m g(x)$. We recall that $Y_1 g(x) \geq 0$ and (2.8), so that we can conclude that $X_m g(x) \geq 0$ for every $x \in \mathbb{G}$ or $X_m g(x) \leq 0$ for every $x \in \mathbb{G}$.

To achieve the proof of the proposition, it is sufficient to show that $Zg = 0$ for every $Z \in V_l$ with $l = 2, \dots, r$ (where r is the step of \mathbb{G}); in fact, recalling the coordinate expression of canonical vector fields, we have $g = g(x_1, \dots, x_m)$ and the thesis follows from (2.4).

Let $Z \in V_2$. Then

$$Z = \sum_{i_1, i_2 \in \{1, \dots, m\}} \alpha_{i_1, i_2} [X_{i_1}, X_{i_2}] \quad (\alpha_{i_1, i_2} \in \mathbb{R})$$

where $X_{i_1} g = k_{i_1} X_m g$, $X_{i_2} g = k_{i_2} X_m g$ and $X_m g \geq 0$ (or $X_m g \leq 0$).

The case $[X_{i_1}, X_{i_2}]g = 0$ is trivial. In particular, we can assume that, for instance $k_{i_1} \neq 0$. Then we have $X_m g = (k_{i_1})^{-1} X_{i_1} g$ from which

$$X_{i_2} g = k_{i_2} (k_{i_1})^{-1} X_{i_1} g \quad \text{with } X_{i_1} g \geq 0 \text{ or } X_{i_1} g \leq 0. \quad (2.9)$$

On the other hand, by hypothesis $[X_j, [X_j, X_i]] = 0$ (with $i, j = 1, \dots, m$), so that

$$\text{span}\{X_{i_1}, X_{i_2} - k_{i_2} (k_{i_1})^{-1} X_{i_1}, [X_{i_1}, X_{i_2}]\}$$

is a Lie algebra isomorphic to the Lie algebra of the Heisenberg group \mathbb{H}^1 (remember $[X_{i_1}, X_{i_2}] \neq 0$). Indicate with \mathfrak{h} the Lie algebra of \mathbb{H}^1 . The following claim, shown inside the proof of Lemma 3.6 in [11], holds:

- if $\tilde{X}_1, \tilde{X}_2 \in \mathfrak{g}$ and $\tilde{X}_1 g \geq 0$, $\tilde{X}_2 g = 0$ and $\tilde{\mathfrak{g}} := \text{span}\{\tilde{X}_1, \tilde{X}_2, [\tilde{X}_1, \tilde{X}_2]\}$ is a subalgebra of \mathfrak{g} isomorphic to \mathfrak{h} , then $[\tilde{X}_1, \tilde{X}_2]g = 0$.

Alternatively this claim can be seen as an easy consequence of Remark 4.9 of [3]. Recalling (2.9), we can conclude that $[X_{i_1}, X_{i_2}]g = 0$. Thus $Zg = 0$ for every $Z \in V_2$.

Now, in order to deal with vector fields belonging to V_3 , we use the basis (Y_1, \dots, Y_m) of V_1 . Since it is a basis, for every $W \in V_3$ there exist $Z_j \in V_2$ such that $W = \sum_{i,j} \beta_{ij} [Y_i, Z_j]$. First of all, we have

$$\text{if } i = 2, \dots, m \text{ then } [Y_i, Z_j]g = 0, \text{ since } Z_j g = 0 \text{ e } Y_i g = 0. \quad (2.10)$$

Consider now the case $i = 1$. We notice that Z_j can be written as a linear combination of two types of commutators: $[Y_1, Y_k]$ and $[Y_l, Y_i]$, with $l \neq 1$ and $i \neq 1$.

Hence, dropping the index j from Z_j , we get

$$[Y_1, Z] = \sum_k \gamma_k [Y_1, [Y_1, Y_k]] + \sum_{l \neq 1, i \neq 1} \lambda_{li} [Y_1, [Y_l, Y_i]].$$

Moreover, by Lemma 2.8,

$$[Y_1, [Y_1, Y_k]] = \sum_{j \neq 1} \mu_{kij} [Y_j, [Y_j, Y_i]] + \sum_{l \neq j, l \neq i} \theta_{kijl} [Y_l, [Y_j, Y_i]].$$

Thus, $[Y_1, Z]g$ can be written as a linear combination of three types of functions:

- i) $[Y_j, Z]g$ with $j \neq 1$ and $Z \in V_2$;
- ii) $[Y_l, [Y_j, Y_i]]g$ with $l \neq j, l \neq i$;
- iii) $[Y_1, [Y_l, Y_i]]g$ with $l \neq 1, i \neq 1$.

By (2.10) above, terms of type i) vanish. Analogously, terms of type iii) vanish since they can be reduced, by Jacobi identity, to a sum of terms of type i). Finally, as for terms of type ii), either $l > 1$ or $l = 1$. If $l > 1$, they vanish again by (2.10). If $l = 1$, then necessarily $j, i \neq 1$ and then, again by Jacobi identity, they can be written as a sum of two terms of type i).

Therefore, $Wg = 0$ for every $W \in V_3$.

In order to complete the proof for the other layers, we show that for every $k \geq 3$ the following claim holds:

$$\begin{aligned} &\text{for every } W \in V_k \text{ there exist } Z_j \in V_{k-1}, \tilde{Z}_r \in V_h \text{ and} \\ &\hat{Z}_s \in V_{k-h} \text{ with } 2 \leq h \leq k-2 \text{ such that } W \text{ is a linear combination} \\ &\text{of commutators of the form } [Y_l, Z_j] \text{ with } l > 1 \text{ and } [\tilde{Z}_r, \hat{Z}_s]. \end{aligned} \quad (2.11)$$

We argue by induction on k . We have just seen the case $k = 3$. Suppose now $n \geq 4$, and assume the claim is true for $k \leq n-1$. We show it holds for $k = n$.

Indeed, for every $W \in V_n$ there exist $Z_i \in V_{n-1}$ such that W is a linear combination of commutators $[Y_j, Z_i]$ with $j = 1, \dots, m$. Obviously, the only commutators to work on in order to show (2.11) are those with $j = 1$.

But every $Z_i \in V_{n-1}$ is a linear combination of commutators of type $[Y_l, \tilde{Z}]$ (with $\tilde{Z} \in V_{n-2}$ and $l > 1$) and $[Z_1, Z_2]$ (with $Z_1 \in V_h$, $Z_2 \in V_{n-1-h}$, $2 \leq h \leq n-3$), we can reduce $[Y_1, Z_i]$ to a linear combination of commutators of type

$$[Y_1, [Y_l, \tilde{Z}]] = -[\tilde{Z}, [Y_1, Y_l]] - [Y_l, [\tilde{Z}, Y_1]]$$

and

$$[Y_1, [Z_1, Z_2]] = -[Z_1, [Z_2, Y_1]] - [Z_2, [Y_1, Z_1]]$$

Hence (2.11) holds for every $k \geq 3$.

Consequently, $Zg = 0$ for every $Z \in V_k$, with $k \geq 3$. Therefore the proposition is proved. \square

Now we can extend Theorem 1.23 to our setting. The proof is the same as in [11], but a more general value of θ_∞ is provided.

Theorem 2.10 (Rectifiability Theorem). *Let \mathbb{G} be a Carnot group of type \star . If $E \subset \mathbb{G}$ is a \mathbb{G} -Caccioppoli set, then*

$$\partial_{\mathbb{G}}^* E \text{ is } (Q-1)\text{-dimensional } \mathbb{G}\text{-rectifiable,} \quad (i)$$

that is $\partial_{\mathbb{G}}^* E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $\mathcal{H}_c^{Q-1}(N) = 0$ and K_h is a compact subset of a \mathbb{G} -regular hypersurface S_h ;

$$\nu_E(p) \text{ is the } \mathbb{G}\text{-normal to } S_h \text{ in } p, \text{ for every } p \in K_h; \quad (ii)$$

$$|\partial E|_{\mathbb{G}} = \theta_c \mathcal{S}_c^{Q-1} \llcorner \partial_{\mathbb{G}}^* E, \quad (iii)$$

where

$$\theta_c(x) = \frac{1}{\omega_{Q-1}} \mathcal{H}^{n-1} \left(\partial S_{\mathbb{G}}^+(\nu_E(x)) \cap U_c(0, 1) \right).$$

ω_k is the k -dimensional measure of the k -dimensional ball in \mathbb{R}^k . If we replace the \mathcal{S}_c -measure with the \mathcal{S}_{∞} -measure, the corresponding density θ_{∞} is a constant. Precisely

$$|\partial E|_{\mathbb{G}} = \theta_{\infty} \mathcal{S}_{\infty}^{Q-1} \llcorner \partial_{\mathbb{G}}^* E, \quad (iv)$$

where

$$\theta_{\infty} = \frac{\omega_{m_1-1} \omega_{m_2} \varepsilon_2^{m_2} \dots \omega_{m_k} \varepsilon_k^{m_k}}{\omega_{Q-1}} = \frac{1}{\omega_{Q-1}} \mathcal{H}^{n-1} \left(\partial S_{\mathbb{G}}^+(\nu_E(0)) \cap U_{\infty}(0, 1) \right).$$

Here ε_i are constants that appears in (1.5) and k is the step of \mathbb{G} .

The following propositions can be proved following the same arguments used in [11].

Theorem 2.11 (Divergence Theorem). *Let \mathbb{G} be a Carnot group of type \star . If $E \subset \mathbb{G}$ is a \mathbb{G} -Caccioppoli set, then*

$$|\partial E|_{\mathbb{G}} = \theta_{\infty} \mathcal{S}_{\infty}^{Q-1} \llcorner \partial_{\star, \mathbb{G}} E, \quad (i)$$

and the following version of the divergence theorem holds

$$- \int_E \operatorname{div}_{\mathbb{G}} \phi \, d\mathcal{L}^n = \theta_{\infty} \int_{\partial_{\star, \mathbb{G}} E} \langle \nu_E, \phi \rangle \, d\mathcal{S}_{\infty}^{Q-1}, \quad \forall \phi \in \mathbf{C}_0^1(\mathbb{G}, H\mathbb{G}). \quad (ii)$$

Corollary 2.12. *If \mathbb{G} is a Carnot group of type \star and a measurable set $E \subset \mathbb{G}$ has boundary of class \mathbf{C}^1 (and hence E is a \mathbb{G} -Caccioppoli set), then*

$$|\partial E|_{\mathbb{G}} = \theta_{\infty} \mathcal{S}_{\infty}^{Q-1} \llcorner \partial E = \left(\sum_{j=1}^{m_1} \langle X_j, n_E \rangle_{\mathbb{R}^n}^2 \right)^{1/2} \mathcal{H}^{n-1} \llcorner \partial E. \quad (i)$$

where n_E denotes the Euclidean outward normal to ∂E . Again a version of the divergence theorem holds

$$- \int_E \operatorname{div}_{\mathbb{G}} \phi \, dx = \theta_{\infty} \int_{\partial E} \langle \nu_E, \phi \rangle \, d\mathcal{S}_{\infty}^{Q-1} \quad \forall \phi \in \mathbf{C}_0^1(\mathbb{G}, H\mathbb{G}). \quad (ii)$$

3. A PROPERTY OF CARNOT GROUPS OF TYPE \star

We do not know whether there exists an alternative characterization of stratified Lie algebra of type \star . However, the following proposition gives a sufficient condition that, if satisfied, yields that a Carnot algebra \mathfrak{g} is *not* of type \star , proving better insights into this condition. In particular, it follows from Proposition 3.1 that free Lie algebras of step greater than 2 and filiform stratified Lie algebras of step greater than 2 are not of type \star .

Unfortunately, the condition is only sufficient: see Example 3.2.

Proposition 3.1. *Let $\mathfrak{g} = V_1 \oplus \dots \oplus V_r$ be a stratified Lie algebra. We set $W_3 := \{[X_k, [X_j, X_i]] \mid i, j, k = 1, \dots, m\}$, so that $V_3 = \text{span}(W_3)$. The following three properties are equivalent:*

(i) *there exists a basis (X_1, \dots, X_m) for V_1 such that*

$$\text{span}(W_3 \setminus \{[X_1, [X_1, X_2]], [X_1, [X_2, X_1]]\}) \subsetneq V_3,$$

i.e. $[X_1, [X_1, X_2]] = -[X_1, [X_2, X_1]]$ is independent of the other commutators;

(ii) *one of the stratified quotient algebras of \mathfrak{g} is the Engel algebra;*

(iii) *one of the stratified quotient algebras of \mathfrak{g} is filiform of step greater than 2;*

Moreover, if \mathfrak{g} satisfies one of these properties, then it is not of type \star ; in fact (i) is incompatible with Lemma 2.8.

Proof. Let us show that (i) \implies (ii). We take the smallest ideal I that contains V_i for $i = 4, \dots, r$, X_k for $k = 3, \dots, m$ and $[X_2, [X_2, X_1]]$. Obviously the step of \mathfrak{g}/I is at most 3 and the dimension of its first layer is 2. We note that $[X_1, X_2], [X_1, [X_1, X_2]] \notin I$, since $[X_1, [X_1, X_2]] = -[X_1, [X_2, X_1]]$ is independent of the other commutators by hypothesis. If we denote with π the canonical projection, we have that $\pi([X_2, [X_2, X_1]]) = 0$ and $\pi([X_1, [X_1, X_2]]) \neq 0$, therefore \mathfrak{g}/I is the Engel algebra.

It's trivial to see that (ii) \implies (iii).

Let us see that (iii) \implies (i). We denote with \mathfrak{f} a stratified quotient algebra of \mathfrak{g} that is filiform of step greater than 2 and we take a basis (Y_1, Y_2) of its first layer \tilde{V}_1 . Since it is filiform, the dimension of its third layer is 1. Hence there exist $(a, b) \neq (0, 0)$ in \mathbb{R}^2 such that

$$a[Y_1, [Y_1, Y_2]] + b[Y_2, [Y_2, Y_1]] = 0 \quad (3.1)$$

If $a = 0$ it becomes $[Y_2, [Y_2, Y_1]] = 0$, whereas if $b = 0$ it can be reduced to the same case by exchanging the role of Y_1 and Y_2 . If $a \neq 0$ and $b \neq 0$, then we perform the change of basis

$$\begin{cases} Y_1 = b\tilde{Y}_1 \\ Y_2 = a\tilde{Y}_1 + \tilde{Y}_2 \end{cases} \quad (3.2)$$

By replacing (3.2) in (3.1), we obtain $[\tilde{Y}_2, [\tilde{Y}_2, \tilde{Y}_1]] = 0$. Therefore, up to a change of basis, $[Y_2, [Y_2, Y_1]] = 0$.

Now we consider the canonical projection $\mathfrak{g} \xrightarrow{\pi} \mathfrak{f}$, which is a surjective homogeneous homomorphism. Hence, there exist two independent vectors X_1, X_2 in V_1 such that $\pi(X_1) = Y_1$ and $\pi(X_2) = Y_2$. We make a basis for V_1 that contains X_1 and X_2 and such that its other elements X_i with $i = 3, \dots, m$ are taken from a

basis of $\text{Ker } \pi|_{V_1}$. This can be done because of linear algebra arguments applied to the surjective linear function $\pi|_{V_1} : V_1 \rightarrow \tilde{V}_1$.

We have that

$$\text{span}(W_3 \setminus \{[X_1, [X_1, X_2]], [X_1, [X_2, X_1]]\}) \subsetneq V_3,$$

otherwise $[Y_1, [Y_1, Y_2]]$ would be zero and \mathfrak{f} would be of step 2, which is a contradiction. \square

Example 3.2. Denote by $\mathfrak{f}_{m,r}$ the free Lie algebra of step r with m generators, and take a free Lie algebra $\mathfrak{f}_{3,r}$ with $r \geq 3$ and three generators X_1, X_2, X_3 . We quotient it by the homogeneous ideal generated by

$$\begin{aligned} & \{[X_1, [X_1, X_2]] + [X_1, [X_1, X_3]], [X_1, [X_1, X_3]] + [X_2, [X_2, X_1]], \\ & [X_2, [X_2, X_1]] + [X_2, [X_2, X_3]], [X_2, [X_2, X_3]] + [X_3, [X_3, X_1]], \\ & [X_3, [X_3, X_1]] + [X_3, [X_3, X_2]]\}, \end{aligned} \quad (3.3)$$

and we denote the obtained stratified quotient algebra by \mathfrak{g} . We stress that vector fields of (3.3) are linearly independent, since Jacobi identity is trivial in those cases. By recalling Example 2.6, we can say that the dimension of the third layer of $\mathfrak{f}_{3,r}$ is 8, whereas the dimension of the third layer of \mathfrak{g} is 3 because of (3.3). In a stratified Lie algebra of type \star with 3 generators, the dimension of the third layer is at most 2, hence the stratified quotient algebra is not of type \star . Now we show by contradiction that (i) of Proposition 3.1 does not hold.

We assume there exists a basis of the first layer of \mathfrak{g} , denoted by (Y_1, Y_2, Y_3) , such that

$$[Y_1, [Y_1, Y_2]] = -[Y_1, [Y_2, Y_1]]$$

is independent of the other commutators. If we rewrite the vectors that generate the ideal as linear combinations of commutators of Y_1, Y_2, Y_3 , we obtain

$$\begin{aligned} & \left\{ \alpha_1[Y_1, [Y_1, Y_2]] + \sum_i \beta_{1i}Z_i, \alpha_2[Y_1, [Y_1, Y_2]] + \sum_i \beta_{2i}Z_i, \right. \\ & \alpha_3[Y_1, [Y_1, Y_2]] + \sum_i \beta_{3i}Z_i, \alpha_4[Y_1, [Y_1, Y_2]] + \sum_i \beta_{4i}Z_i, \\ & \left. \alpha_5[Y_1, [Y_1, Y_2]] + \sum_i \beta_{5i}Z_i \right\}, \end{aligned} \quad (3.4)$$

where Z_i are the remaining commutators of length 3, excluding $[Y_1, [Y_2, Y_1]]$. We remark that, since our assumptions,

$$\alpha_i = 0 \quad \text{for } i = 1, \dots, 5. \quad (3.5)$$

If

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

is the relation between the two basis, we can replace coefficients α_i in (3.5) with their expression and we obtain

$$\begin{cases} a_{11}(-a_{12}a_{21} + a_{11}a_{22}) + a_{11}(-a_{12}a_{31} + a_{11}a_{32}) = 0 \\ -a_{21}(-a_{12}a_{21} + a_{11}a_{22}) + a_{11}(-a_{12}a_{31} + a_{11}a_{32}) = 0 \\ -a_{21}(-a_{12}a_{21} + a_{11}a_{22}) + a_{21}(-a_{22}a_{31} + a_{21}a_{32}) = 0 \\ -a_{31}(-a_{12}a_{31} + a_{11}a_{32}) + a_{21}(-a_{22}a_{31} + a_{21}a_{32}) = 0 \\ -a_{31}(-a_{12}a_{31} + a_{11}a_{32}) - a_{31}(-a_{22}a_{31} + a_{21}a_{32}) = 0. \end{cases}$$

This system of equations leads to the following solutions:

$$a_{11} = 0 \text{ and } a_{21} = 0 \text{ and } a_{31} = 0$$

or

$$a_{12} = 0 \text{ and } a_{22} = 0 \text{ and } a_{32} = 0$$

or

$$a_{11} = 0 \text{ and } a_{12} = 0 \text{ and } a_{21} = 0 \text{ and } a_{22} = 0$$

or

$$a_{11} = 0 \text{ and } a_{12} = 0 \text{ and } a_{31} = 0 \text{ and } a_{32} = 0$$

or

$$a_{21} = 0 \text{ and } a_{22} = 0 \text{ and } a_{31} = 0 \text{ and } a_{32} = 0$$

or

$$a_{21} = 0 \text{ and } a_{22} = 0 \text{ and } a_{11} = \frac{a_{12}a_{31}}{a_{32}} \text{ and } a_{32} \neq 0$$

or

$$a_{11} = \frac{a_{12}a_{21}}{a_{22}} \text{ and } a_{22} \neq 0 \text{ and } a_{31} = 0 \text{ and } a_{32} = 0$$

or

$$a_{11} = \frac{a_{12}a_{21}}{a_{22}} \text{ and } a_{22} \neq 0 \text{ and } a_{21} = \frac{a_{22}a_{31}}{a_{32}} \text{ and } a_{32} \neq 0$$

or

$$a_{11} = 0 \text{ and } a_{12} = 0 \text{ and } a_{21} \neq 0 \text{ and } a_{21} = \frac{a_{22}a_{31}}{a_{32}} \text{ and } a_{32} \neq 0.$$

In any of these cases, the change of basis matrix is singular, which is a contradiction.

4. EXAMPLES OF CARNOT GROUPS FOR WHICH BLOW-UP DOES NOT HOLD

Example 4.1. Let \mathbb{G} be a free Carnot group of step $r > 2$ with m generators ($m \geq 2$). Then Blow-up Theorem does not hold.

Theorem 14.1.10 of [4] gives a model for its Lie algebra \mathfrak{g} in terms of m generating vector fields with polynomial coefficients on \mathbb{R}^n , where n is the dimension of \mathfrak{g} . By Remark 14.1.11 of [4], these m vector fields naturally define a free Carnot group of step r and m generators: more precisely, they are left invariant vector fields of a Carnot group $(\mathbb{R}^n, \circ, \delta_\lambda)$ that is isomorphic to \mathbb{G} . In general, the coordinate system given by this isomorphism is not the exponential one defined in 1.2. We denote these generating vector fields with X_1, X_2, \dots, X_m .

By Theorem 14.1.10 of [4], we have that

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + \dots + \frac{x_1^2}{2} \frac{\partial}{\partial x_k} + \dots + a_{2,n}(x_1, x_2, \dots, x_n) \frac{\partial}{\partial x_n},$$

where k represent the position of $[[X_2, X_1], X_1]$ in the Hall basis for $\mathfrak{f}_{m,r}$. Moreover, in X_3, \dots, X_m , the partial derivative $\frac{\partial}{\partial x_k}$ does not appear.

Let $E = \{x \in \mathbb{G} : f(x) \geq 0\}$, where

$$f(x_1, \dots, x_n) = \frac{x_2^3}{3} + 2x_k.$$

We note that $\partial E = \{x \in \mathbb{R}^4 : f(x) = 0\}$ is a smooth Euclidean manifold, hence E is a \mathbb{G} -Caccioppoli set (see [11], Proposition 2.22). We stress that ∂E is not a vertical hyperplane of \mathbb{G} .

The horizontal gradient of f is $\nabla_{\mathbb{G}} f(x) = (0, x_1^2 + x_2^2)$ and the generalized inward \mathbb{G} -normal is

$$\nu_E(x) = -\frac{\nabla_{\mathbb{G}} f(x)}{|\nabla_{\mathbb{G}} f(x)|} = (0, -1)$$

for every $x \in \partial E \setminus N$, where $N = \{x \in \mathbb{E} : x_1 = x_2 = 0\}$. Since $|\partial E|_{\mathbb{G}}(N) = 0$, the origin belongs to $\partial_{\mathbb{G}}^* E$. We note that $f(\delta_{\lambda} x) = \lambda^3 f(x)$ for $\lambda > 0$, hence $E_{\lambda,0} = \delta_{\lambda} E = E$. Finally we can conclude that (1.12) is false since E is not a vertical halfspace.

Example 4.2. Let \mathbb{G} be a filiform Carnot group whose Lie algebra is of type

$$\begin{aligned} & \text{span}\{X_1, X_2\} \oplus \text{span}\{[X_2, X_1]\} \oplus \text{span}\{[[X_2, X_1], X_1]\} \oplus \dots \\ & \dots \oplus \text{span}\{[\underbrace{\dots [X_2, X_1], X_1], \dots}_{(r-1) \text{ times}}, X_1]\} \end{aligned}$$

where $r > 2$ and all other independent commutators are identically zero. Then Blow-up Theorem does not hold. A coordinate expression of vector fields X_1 and X_2 is

$$X_1 = \frac{\partial}{\partial x_1} \quad X_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + \frac{x_1^2}{2} \frac{\partial}{\partial x_4} + \dots + (-1)^{(r-1)} \frac{x_1^{r-1}}{(r-1)!} \frac{\partial}{\partial x_{r+1}}$$

Let $E = \{x \in \mathbb{G} : f(x) \geq 0\}$, where

$$f(x_1, \dots, x_{r+1}) = \frac{x_2^3}{3} + 2x_4.$$

Then the proof follows the same argument used in the previous example.

REFERENCES

- [1] Ambrosio, L.: Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces, *Adv. in Math.* **159**, 51–67 (2001)
- [2] Ambrosio, L.: La teoria dei perimetri di Caccioppoli–De Giorgi e i suoi più recenti sviluppi. *Rend. Lincei Mat. Appl.* **21**, 275–286 (2010)
- [3] Ambrosio, L., Kleiner, B., Le Donne, E.: Rectifiability of Sets of Finite Perimeter in Carnot Groups: Existence of a Tangent Hyperplane. *J. Geom. Anal.* **19**, 509–540 (2009)
- [4] Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*. Springer, New York (2007)
- [5] De Giorgi, E.: Nuovi teoremi relativi alle misure $(r-1)$ -dimensionali in uno spazio ad r dimensioni. *Ricerche Mat.* **4**, 95–113 (1955)
- [6] Evans, L.C., Gariepy, R.F.: *Measure Theory and Fine Properties of Functions*. CRC Press, BocaRaton (1992).

- [7] Eves, H.W.: Elementary matrix theory. Courier Dover Publications, New York (1980)
- [8] Folland, G.B.: Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.* **13**, 161–207 (1975)
- [9] Folland, G.B., Stein, E.M.: Hardy spaces on homogeneous groups. Princeton University Press, Princeton (1982)
- [10] Franchi, B., Serapioni, R., Serra Cassano, F.: Rectifiability and perimeter in the Heisenberg group. *Math Ann* **321**, 479–531 (2001)
- [11] Franchi, B., Serapioni, R., Serra Cassano, F.: On the Structure of Finite Perimeter Sets in Step 2 Carnot Groups. *J. Geom. Anal.* **13**, 421–466 (2003)
- [12] Franchi, B., Serapioni, R., Serra Cassano, F.: Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields. *Houston J. Math.* **22**(4), 859–889 (1996)
- [13] Franchi, B., Serapioni, R., Serra Cassano, F.: Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups. *Comm. Anal. Geom.* **11**(5), 909–944 (2003)
- [14] Garofalo, N., Nhieu, D.M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.* **49**, 1081–1144 (1996)
- [15] Gorbatsevich, V.V., Onishchik, A.L., Vinberg, E.B.: Foundations of Lie Theory and Lie Transformation Groups. Springer, Berlin (1997)
- [16] Korányi, A., Reimann, H.M.: Foundation for the theory of quasiconformal mappings on the Heisenberg group. *Advances in Mathematics* **111**, 1–87 (1995)
- [17] Magnani, V.: Characteristic points, rectifiability and perimeter measure on stratified groups. *J. Eur. Math. Soc.* **8**(4), 585–609 (2006)
- [18] Miranda, M.Jr.: Functions of bounded variation on “good” metric measure spaces. *J. Math. Pures Appl.* (9) **82**, no.8, 975–1004 (2003)
- [19] Mitchell, J.: On Carnot-Carathéodory metrics. *J. Differ. Geom.* **21**, 35–45 (1985)
- [20] Montgomery, R.: A Tour of Subriemannian Geometries, Their Geodesics and Applications. *Mathematical Surveys and Monographs* **91**, AMS, Providence RI (2002)
- [21] Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields I: Basic properties. *Acta Mathematica* **155**(1), 103–147 (1985)
- [22] Pansu, P.: Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math.* **129**, 1–60 (1989)
- [23] Varopoulos, N.Th., Saloff-Coste, L., Coulhon, T.: Analysis and Geometry on Groups. Cambridge University Press, Cambridge (1992)
- [24] Ziemer, W.P.: Weakly Differentiable Functions. Springer-Verlag, New York (1989).

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